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Homogenization of non-periodic masonry structures

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Abstract

A homogenization approach to assess the mechanical characteristics of masonry structures is presented in this paper. In order to analyze an actual masonry, the concept of periodic cell, used in literature for periodic masonry, is replaced with that of representative volume element. This volume is found by employing a formulation based on finite size test-windows. The homogenized medium stiffness tensor is obtained by considering the hierarchy of estimates relative to essential and natural boundary conditions. Moreover an ensemble average is performed on space taking into account different test-windows location on the given structure. An application shows the effectiveness of the proposed approach.

Keywords: Elastic moduli; Homogenization; Masonry-like materials; Non-homogeneous media

1. Introduction

The analysis of the masonry structures has received great interest in last decades. This is partly related to the application of brickwork for new structures. In fact, the main interesting aspect is linked to analyzing historical and monumental buildings made of masonry material (stone masonry or brickwork).

In literature, two different approaches have been proposed: the discrete models and the continuous models.

Employing the finite element technique, the discrete models describe the masonry by using the actual physical and geometrical properties of block units and mortar joints (Page, 1978). However, these approaches have serious limitations relative to ill-conditioned and/or non-stable numerical solutions and to the impracticability in the context of large-scale masonry structures as stated by Pietruszczak and Niu (1992).

In the case of continuous models, the behavior of masonry has been described by phenomenological law (Heymann, 1966; Giaquinta and Giusti, 1985) or in the framework of micromechanics theory.

In the latter, the masonry is modeled as a heterogeneous material composed by bricks in a matrix of mortar. The homogenization techniques allow to define a homogeneous body in order to study the linear and non-linear behavior of the masonry.

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Nevertheless, the techniques proposed in literature adopt the hypothesis of "periodic-structure" for the masonry. This involves assuming bricks, head and bed mortar joints of equal dimensions and characteristics. Moreover, these components must be arranged in a periodic pattern. However, this hypothesis can be accepted for new structures only. The periodic approach is surely incorrect for a very large number of existing masonry structures, which vice versa have a great cultural and social interest such as in maintaining and restoring historical and monumental buildings. In order to apply the homogenization theory to old masonry, a different approach is necessary.

This aspect is the main topic here. The paper is written in the following sequence. First, the main results in literature about homogenization of masonry are briefly illustrated, underlining the periodicity hypothesis. Then the fundamentals of the mechanics of non-periodic composites are reviewed. This will allow to introduce the proposed homogenization method based on "test-windows". The application of the method is showed by a numerical example considering an actual masonry wall.

2. Periodic masonry structures

Given a heterogeneous body made of materials with different properties and geometry, the homogenization technique allows to obtain an "equivalent body" (Sanchez-Palencia and Zaoui, 1987; Suquet, 1982).

The first approach to the homogenization of the masonry material is due to Pande et al. (1989). Assuming continuous head joints and no-slippage between the mortar layers and brick units, the expressions for the elastic properties of the equivalent material were derived in terms of the elastic proprieties of the brick and mortar together with relative thickness. The proposed homogenization procedure is performed in several steps and its result depends on the sequence of the successive steps (Geymonat et al., 1987).

In order to describe the average mechanical properties, Pietruszczak and Niu (1992) proposed to address the influence of head and bed joint separately (concept of a superimposed medium). Considering the head joints, the homogenized medium was regarded as an orthotropic elastic–brittle material and the mechanical properties were determined from Eshelby's (1957) solution to an ellipsoidal inclusion problem combined with Mori–Tanaka's mean-field theory (Mori and Tanaka, 1973). Representing the masonry by a medium stratified with a family of bed joints, which form the weakest link in the microstructure of the system, the average constitutive relation for the entire composite system was obtained from the averaging rule (Hill, 1963). Under simplified hypothesis about microstructure geometry and micro–macro quantities relationships, constitutive laws for the homogenized material were also proposed by Maier et al. (1991).

In order to overcome the previous simplifying hypothesis, Anthoine (1995) proposed a rigorous application of the homogenization theory for the periodic media, based on asymptotic analysis (Bensoussan et al., 1978; Sanchez-Palencia, 1980). Following this approach, the periodicity is characterized by a frame of reference and it is sufficient to define the mechanical characteristics of the media on a small domain (cell) to be repeated by translation. The proposed numerical approach for deriving the global elastic coefficients of masonry takes into account the elasticity of both constituents (brick and mortar) as well as the finite thickness of the joints.

The constitutive behavior of in plane loaded dry block masonry, with attention to failure analysis, was analyzed by Alpa and Monetto (1994).

Using the homogenization technique implemented within the framework of the yield design, an approach to the ultimate strength of brick masonry was proposed by De Buhan and De Felice (1997). In particular, the masonry was described by a regular assemble of homogeneous brick, obeying a plane stress failure, separated by joints considered as one-dimensional interface.

A masonry damage model due to the growth of fractures only in mortar joints was proposed by Luciano and Sacco (1997, 1998a). In this work, the homogenization theory for material with periodic microstructure was used to define the overall moduli of the uncracked and cracked masonry (Luciano and Sacco, 1997, 1998b).

A body exhibiting periodic structure is also introduced by Cecchi and Di Marco (2000) and Cecchi and Sab (2002) to analyze the effect of rigid or elastic blocks in the homogenization of masonry wall.

The previous approaches are based on the periodic structure hypothesis: this involves assuming masonry formed by regular blocks, with fixed dimensions, and interposed bed and head of mortar. The pattern formed by the arrangement of bricks and bed of mortar is periodic. However, these features can only be followed in a new masonry.

When we analyze an old masonry, we find that it is formed by blocks of different dimensions that are arranged in a non-periodic pattern, with the mortar joints assuming different thickness. In order to apply the homogenization theory to this type of masonry it is necessary to use a different approach.

3. Fundamentals of the mechanics of non-periodic composites

3.1. Representative volume element

Considering a random composite, the concept of periodic cell, is replaced with that of *representative volume element* (RVE): it is defined as a portion of the composite material with the following features (Aboudi, 1991; Christensen, 1980):

1. Structurally, it is entirely typical of the whole composite on average;
2. Contains a sufficient number of material phases.

If the material is under a macroscopically uniform homogeneous state of stress, i.e. all the material portions having the dimensions of the RVE undergo the same loading conditions, then for scales smaller than RVE the actual arrangement of bricks and mortar joints can be considered, whereas for scales greater than RVE, the composite can be replaced with an homogenized material.

The properties of the homogenized material can be determined by analyzing the RVE: in detail, a relation between the average values of stresses and strains is established. These average values are defined as:

$$\begin{aligned} \bar{\varepsilon}_{ij} &= \frac{1}{V}\int_V \varepsilon_{ij}\,\mathrm{d}V \\ \bar{\sigma}_{ij} &= \frac{1}{V}\int_V \sigma_{ij}\,\mathrm{d}V \end{aligned} \tag{1}$$

These properties can be found by applying a different kind of conditions on the boundary of the RVE. The boundary conditions can be applied:

(A) in terms of displacements u_i,

$$u_i = \varepsilon^0_{ij} x_j \tag{2}$$

where ε^0_{ij} are constant strains and x_j are point coordinates, or

(B) in term of tractions t_i,

$$t_i = \sigma^0_{ij} n_j \tag{3}$$

where σ^0_{ij} are constant stresses and n_j are the components of the unit outward normal vector to boundary.

The strains average values in conditions (A) are $\bar{\varepsilon}_{ij} = \varepsilon^0_{ij}$ while stresses average values in conditions (B) are $\bar{\sigma}_{ij} = \sigma^0_{ij}$. If it's possible to find the values of average stresses in case (A) and of average strains in case (B), the elastic stiffness constants C^*_{ijkl} (in the latter case the elastic compliances S^*_{ijkl}), are obtained by:

$$\begin{aligned} &(\mathrm{A})\ \bar{\sigma}_{ij} = C^*_{ijkl}\bar{\varepsilon}_{kl} \\ &(\mathrm{B})\ \bar{\varepsilon}_{ij} = S^*_{ijkl}\bar{\sigma}_{kl} \end{aligned} \tag{4}$$

For a material composed by two phases, such as masonry, the following expressions for average stresses and strains are obtained (apex (k) indicates the quantity relative to phase k):

$$\begin{aligned} \bar{\sigma}_{ij} &= c_1\bar{\sigma}^{(1)}_{ij} + c_2\bar{\sigma}^{(2)}_{ij} \\ \bar{\varepsilon}_{ij} &= c_1\bar{\varepsilon}^{(1)}_{ij} + c_2\bar{\varepsilon}^{(2)}_{ij} \end{aligned} \tag{5}$$

where c_1 and c_2 are the fractional concentrations by volume of phases 1 and 2 respectively, and $c_1 + c_2 = 1$. By writing the relations between stresses and strains in the phases as

$$\begin{aligned} \sigma^{(1)}_{ij} &= C^{(1)}_{ijkl}\varepsilon_{kl}; \quad \varepsilon^{(1)}_{ij} = S^{(1)}_{ijkl}\sigma_{kl} \\ \sigma^{(2)}_{ij} &= C^{(2)}_{ijkl}\varepsilon_{kl}; \quad \varepsilon^{(2)}_{ij} = S^{(2)}_{ijkl}\sigma_{kl} \end{aligned} \tag{6}$$

the following relations between the global average stresses and the average strains in the two phases (and the analogous between the global average strains and the average stresses in the two phases) can be written:

$$\begin{aligned} \bar{\sigma}_{ij} &= c_1 C^{(1)}_{ijkl}\bar{\varepsilon}^{(1)}_{kl} + c_2 C^{(2)}_{ijkl}\bar{\varepsilon}^{(2)}_{kl} \\ \bar{\varepsilon}_{ij} &= c_1 S^{(1)}_{ijkl}\bar{\sigma}^{(1)}_{kl} + c_2 S^{(2)}_{ijkl}\bar{\sigma}^{(2)}_{kl} \end{aligned} \tag{7}$$

By writing the relations between the averages on the single phase and the global average as

$$\begin{aligned} \bar{\varepsilon}^{(1)}_{ij} &= A^{(1)}_{ijkl}\bar{\varepsilon}_{kl}; \quad \bar{\varepsilon}^{(2)}_{ij} = A^{(2)}_{ijkl}\bar{\varepsilon}_{kl} \\ \bar{\sigma}^{(1)}_{ij} &= B^{(1)}_{ijkl}\bar{\sigma}_{kl}; \quad \bar{\sigma}^{(2)}_{ij} = B^{(2)}_{ijkl}\bar{\sigma}_{kl} \end{aligned} \tag{8}$$

($\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$, $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$ are concentration matrices, where $c_1\mathbf{A}^{(1)} + c_2\mathbf{A}^{(2)} = \mathbf{I}$ and $c_1\mathbf{B}^{(1)} + c_2\mathbf{B}^{(2)} = \mathbf{I}$, $\mathbf{I}$ being the unit matrix) it is found that

$$\begin{aligned} C^*_{ijkl} &= c_1 C^{(1)}_{ijkl}A^{(1)}_{ijkl} + c_2 C^{(2)}_{ijkl}A^{(2)}_{ijkl} \\ S^*_{ijkl} &= c_1 S^{(1)}_{ijkl}B^{(1)}_{ijkl} + c_2 S^{(2)}_{ijkl}B^{(2)}_{ijkl} \end{aligned} \tag{9}$$

where $\mathbf{C}^*$ and $\mathbf{S}^*$ of the homogenized medium are defined in terms of the stiffness of the individual phases.

Since the above operations are done on RVE, it is found that $\mathbf{C}^* = (\mathbf{S}^*)^{-1}$. Nevertheless, equations from (1)–(9) hold for every portion of heterogeneous material, but in this case it is found that the values of $\mathbf{C}^*$ and $(\mathbf{S}^*)^{-1}$ are different.

By assuming particular values of the concentration matrices, approximate values of the effective stiffness and compliances can be found.

Voight proposed that the strain in the composite be uniform, so $\mathbf{A}^{(1)} = \mathbf{A}^{(2)} = \mathbf{I}$.

This yields:

$$C^{\mathrm{V}}_{ijkl} = c_1 C^{(1)}_{ijkl} + c_2 C^{(2)}_{ijkl} \tag{10}$$

Reuss proposed that the stress in the composite be uniform, so $\mathbf{B}^{(1)} = \mathbf{B}^{(2)} = \mathbf{I}$.

This yields:

$$S^{R}_{ijkl} = c_1 S^{(1)}_{ijkl} + c_2 S^{(2)}_{ijkl} \tag{11}$$

It should be noted that neither of the two assumptions is correct: Voight's violates equilibrium, while Reuss's violates compatibility.

Nevertheless, these two values can still be useful: in fact Hill's theorem demonstrates that these two values set limits to the range where the actual stiffness can be found. Writing Voight's with $\mathbf{C}^{V}$, Reuss's with $\mathbf{C}^{R}$ ($\mathbf{C}^{R} = (\mathbf{S}^{R})^{-1}$) and with $\mathbf{C}$ the actual stiffness, the following chain of inequalities rules:

$$\mathbf{C}^{R} \leqslant \mathbf{C} \leqslant \mathbf{C}^{V} \tag{12}$$

where the order relation $\mathbf{X} \leqslant \mathbf{Y}$ means that $\mathbf{z}^{T}\mathbf{X}\mathbf{z} \leqslant \mathbf{z}^{T}\mathbf{Y}\mathbf{z}$ for any vector $\mathbf{z} \neq \mathbf{0}$. Anyway, this range is too wide, in most cases, to be of practical interest, such closer limits have to be determined. In any case, these limits are still useful to check the correctness of the proposed procedure.

3.2. The solution in terms of boundary conditions

3.2.1. Essential boundary conditions

By applying at RVE the boundary conditions in terms of displacements as given in (2) the following relation between stiffness and average strains can be found:

$$\begin{aligned} C^{e}_{ijkl}\varepsilon^{0}_{kl} &= C^{e}_{ijkl}\bar{\varepsilon}_{kl} = \bar{\sigma}_{ij} = c_1\bar{\sigma}^{(1)}_{ij} + c_2\bar{\sigma}^{(2)}_{ij} = c_1 C^{(1)}_{ijkl}\bar{\varepsilon}^{(1)}_{kl} + c_2 C^{(2)}_{ijkl}\bar{\varepsilon}^{(2)}_{kl} = C^{(1)}_{ijkl}(\varepsilon^{0}_{kl} - c_2\bar{\varepsilon}^{(2)}_{kl}) + c_2 C^{(2)}_{ijkl}\bar{\varepsilon}^{(2)}_{kl} \\ &= C^{(1)}_{ijkl}\varepsilon^{0}_{kl} + c_2(C^{(2)}_{ijkl} - C^{(1)}_{ijkl})\bar{\varepsilon}^{(2)}_{kl} \end{aligned} \tag{13}$$

(note that stiffness is identified with the over script "e", because it is determined under essential conditions, while $\mathbf{C}^{(1)}$ and $\mathbf{C}^{(2)}$ are the actual stiffness of the two phases).

Assuming for $\boldsymbol{\varepsilon}^{0}$ the matrix $\mathbf{I}^{mn}$, where $\mathbf{I}^{mn}$ is the matrix with all the components set at 0 except for the component (mn) which is set at 1, it is found that

$$C^{e}_{ijmn} = C^{(1)}_{ijmn} + c_2(C^{(2)}_{ijkl} - C^{(1)}_{ijkl})\bar{\varepsilon}^{mn(2)}_{kl} \tag{14}$$

where $\bar{\varepsilon}^{mn(2)}_{kl}$ is the average strain in phase 2 when boundary conditions $u_i = I^{mn}_{ij}x_j$ are applied.

If is easier to work with stresses rather than with strains, Eq. (14) is transformed into

$$C^{e}_{ijmn} = C^{(1)}_{ijmn} + c_2(C^{(2)}_{ijkl} - C^{(1)}_{ijkl})S^{(2)}_{klrt}\bar{\sigma}^{mn(2)}_{rt} \tag{15}$$

where $\bar{\sigma}^{mn(2)}_{kl}$ is the average stress in phase 2.

It is worth noting that, in order to determine all the components of $\mathbf{C}^{e}$ we must use 6 (in the case of space problem) or 3 (in the case of plane problem) different matrices $\mathbf{I}^{mn}$, which forms a base for all the possible values of $\boldsymbol{\varepsilon}^{0}$ (i.e. $\boldsymbol{\varepsilon}^{0}$ can be expressed as $\boldsymbol{\varepsilon}^{0} = k_{11}\mathbf{I}^{11} + k_{12}\mathbf{I}^{12} + k_{13}\mathbf{I}^{13} + k_{22}\mathbf{I}^{22} + k_{23}\mathbf{I}^{23} + k_{33}\mathbf{I}^{33}$).

3.2.2. Natural boundary conditions

By applying at the boundary of the test-window conditions in terms of tractions as given in (3) the following relation between stiffness and average stresses can be found:

$$\begin{aligned} S^{n}_{ijkl}\sigma^{0}_{kl} &= S^{n}_{ijkl}\bar{\sigma}_{kl} = \bar{\varepsilon}_{ij} = c_1\bar{\varepsilon}^{(1)}_{ij} + c_2\bar{\varepsilon}^{(2)}_{ij} = c_1 S^{(1)}_{ijkl}\bar{\sigma}^{(1)}_{kl} + c_2 S^{(2)}_{ijkl}\bar{\sigma}^{(2)}_{kl} = S^{(1)}_{ijkl}(\sigma^{0}_{kl} - c_2\bar{\sigma}^{(2)}_{kl}) + c_2 S^{(2)}_{ijkl}\bar{\sigma}^{(2)}_{kl} \\ &= S^{(1)}_{ijkl}\sigma^{0}_{kl} + c_2(S^{(2)}_{ijkl} - S^{(1)}_{ijkl})\bar{\sigma}^{(2)}_{kl} \end{aligned} \tag{16}$$

(note that compliance is identified with the over script "n", because it is determined under natural conditions, while $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ are the actual compliances of the two phases).

Assuming for $\boldsymbol{\sigma}^0$ the matrix $\mathbf{I}^{mn}$ as defined before, it is found that

$$S^{\mathrm{n}}_{ijmn} = S^{(1)}_{ijmn} + c_2(S^{(2)}_{ijkl} - S^{(1)}_{ijkl})\bar{\sigma}^{mn(2)}_{kl} \tag{17}$$

where $\bar{\sigma}^{mn(2)}_{kl}$ is the average stress in phase 2 when boundary conditions $t_i = \sigma^0_{ij} n_j$ are applied.

To find the stiffness $\mathbf{C}^{\mathrm{n}}$ it is necessary to invert the compliance, $\mathbf{C}^{\mathrm{n}} = (\mathbf{S}^{\mathrm{n}})^{-1}$.

4. Homogenization of quasi-periodic masonry—the "test-window" method

It is difficult to identify RVE in old masonry. In fact, for a periodic structure, a periodic cell which, by opportune translations, generates the whole structure, is used (Anthoine, 1995). In composite material such as fiber-reinforced ones, it is sufficient to choose a relatively large portion of material, depending on the size of the inclusions (Ostoja-Starzewski, 1998), to determine the RVE. The definition of the material statistical homogeneity and the proof of convergence to RVE as the material sample size approaches infinity are addressed in Huet (1990) and in Sab (1992).

Vice versa, in old masonry the inclusions (the bricks or stones) in the matrix (the mortar) can assume different relative dimensions and arrangement, depending on the portion of masonry analyzed. So the masonry can be defined as a "quasi-periodic" material, and the identification of the RVE is not possible a priori (Fig. 1).

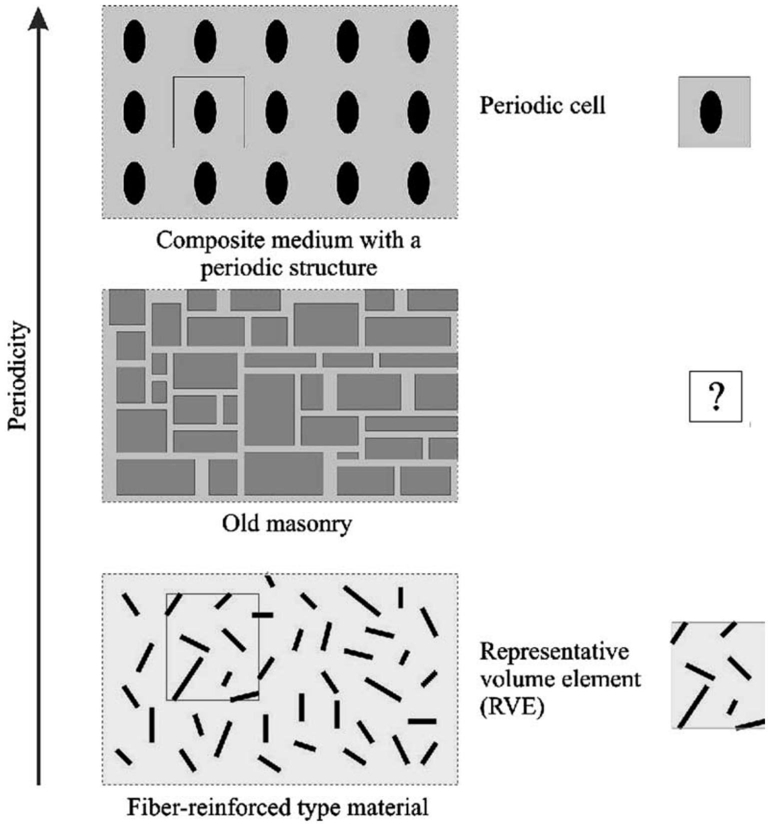

Fig. 1. RVE identification versus material periodicity.

The method here developed to find the RVE is an iterative one:

1. the first step is to choose the dimensions of a rectangle, the test-window;
2. the test-window is then placed inside masonry and a portion of material (which contains both phases) is analyzed; it should be noted that the position of the test-window is arbitrary;
3. the next step is to determine the values of the stiffness by assuming boundary conditions in terms of displacements, or essential conditions (14) and (15), and in terms of tractions, or natural conditions (17). The values obtained using essential conditions differ from those obtained under natural conditions, and it has been found that $\mathbf{C}^{\mathrm{n}} \leqslant \mathbf{C}^{\mathrm{e}}$ in agreement with Huet (1990) and Sab (1992);
4. steps 2 and 3 are repeated for different positions of the test-window inside the masonry;
5. then, the average of the $\mathbf{C}^{\mathrm{n}}$ and $\mathbf{C}^{\mathrm{e}}$ found for the different positions is calculated, and the values $\langle \mathbf{C}^{\mathrm{n}} \rangle$ and $\langle \mathbf{C}^{\mathrm{e}} \rangle$ are obtained;
6. the effective elastic stiffness components are estimated by:

$$\widetilde{C}^{*}_{ij} = \frac{\langle C^{\mathrm{n}} \rangle_{ij} + \langle C^{\mathrm{e}} \rangle_{ij}}{2} \tag{18}$$

The range amplitude δ is defined as

$$\delta = \max_{ij} \left| \frac{\langle C^{\mathrm{e}} \rangle_{ij} - \langle C^{\mathrm{n}} \rangle_{ij}}{\widetilde{C}^{*}_{ij}} \right| \tag{19}$$

7. the next step is to choose new increased dimensions for the test-window, and to repeat steps from 2 to 6;
8. the iteration is stopped when δ is sufficiently small, which is to say that $\langle \mathbf{C}^{\mathrm{n}} \rangle$ and $\langle \mathbf{C}^{\mathrm{e}} \rangle$ are sufficiently close.

It should be noted that, in general, $\langle \mathbf{C}^{\mathrm{n}} \rangle$ and $\langle \mathbf{C}^{\mathrm{e}} \rangle$ do not define a range in which the effective value of stiffness $\mathbf{C}^{*}$ can be found. As the dimensions of the test-window increase, $\langle \mathbf{C}^{\mathrm{n}} \rangle$ and $\langle \mathbf{C}^{\mathrm{e}} \rangle$ converge to $\mathbf{C}^{*}$, which can be estimated by $\widetilde{\mathbf{C}}^{*}$.

The test-window is placed in different positions inside the masonry to overcome the quasi-periodicity of the material: in fact, both in the case of periodic and non-periodic material, the position on the window does not affect the result, and one could concentrate only on the variation of the dimensions of the window.

5. Application

The proposed method was applied to determine the stiffness of an actual masonry wall made up of stones arranged with bed and head of mortar (Fig. 2(a)). The method is applied under the hypothesis of plane stress. In this way, the problem is completely defined by the three components of stress σ_{11}, σ_{22} and σ_{12} and by the three components of strain ε_{11}, ε_{22} and ε_{12}.

The masonry object of the study has the following features:

(a) it has an irregular pattern, i.e. the constitutive blocks of stone have different dimensions, although all the blocks belonging to the same row are about the same height; furthermore, the dimensions do not change very much when different rows are considered;
(b) the shape of the stones is quite rectangular: it is worth noting that the roundness at the corner of the brick does not affect the stiffness of the composite in a significant way (Cuomo and Perticone, 1996), but it increases the difficulties in modeling the masonry.

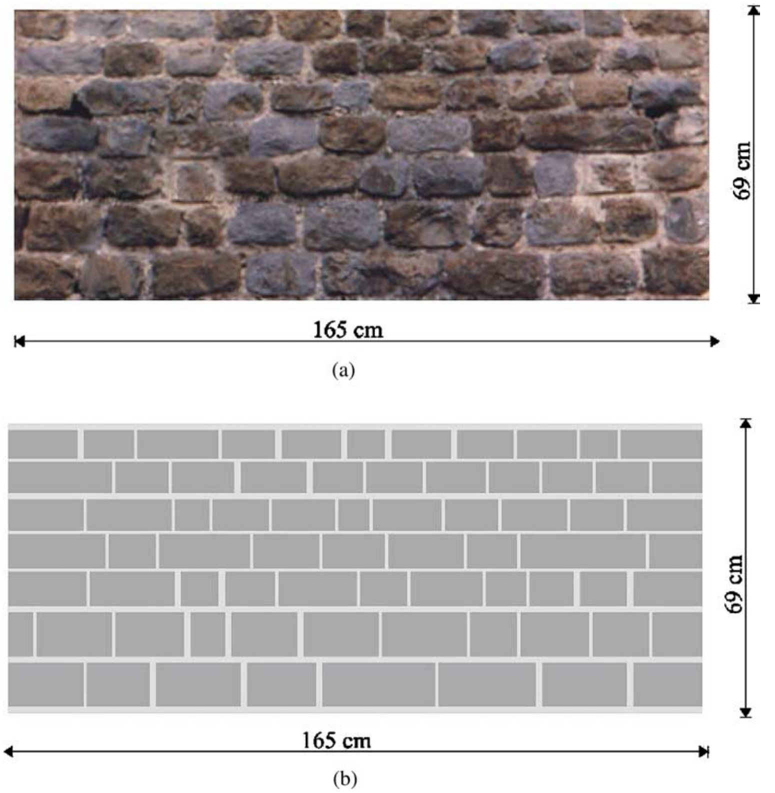

Fig. 2. (a) The masonry wall; (b) Simplified geometrical pattern.

Before applying the test-window method, a few steps are necessary to set up the model: the masonry is first geometrically simplified, i.e. all the block are assumed rectangular and the thickness of the bed of mortar constant (Fig. 2(b)).

To solve the problem in terms of essential and natural conditions, the FEM is used. The components of the masonry are assumed to have the elastic properties reported in Table 1.

At first the test-window has small dimensions, but still quite large to contain both stone and mortar elements. The window is then placed in various positions inside the masonry (four in the present application, as in Fig. 3(a)).

The portion of masonry enclosed by any window, with its arrangement of stone and mortar, is then extracted from the whole and analyzed alone. This portion is modeled with a mesh of finite elements of two

Table 1
Mechanical properties of the masonry phases

Phase	Material	Young's module, E (MPa)	Poisson's coefficient, ν	Stiffness components (MPa)			
				C_{11}	C_{12}	C_{22}	C_{33}
1	Stone	12,500	0.20	13,021	2604	13,021	10,417
2	Mortar	1200	0.30	1319	396	1319	923

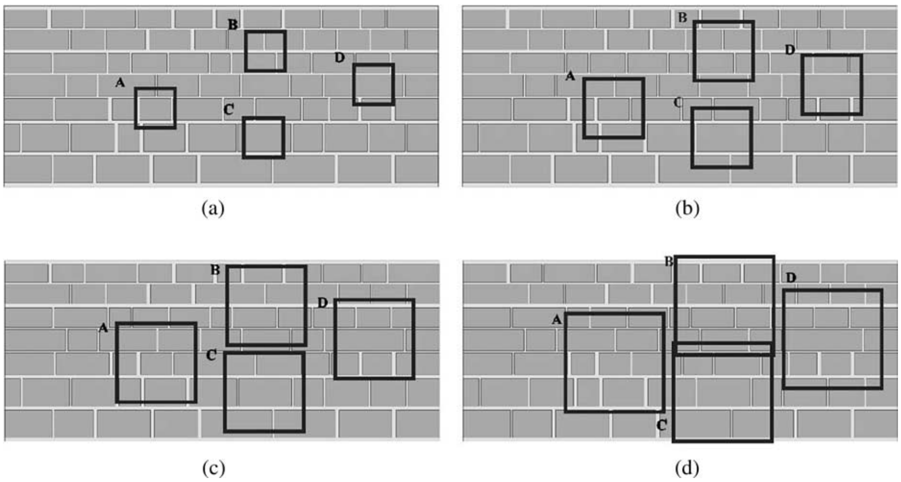

Fig. 3. Position of the test-windows inside the masonry wall: (a) test-windows dimensions 150×150 mm; (b) 225×225 mm; (c) 300×300 mm; (d) 375×375 mm.

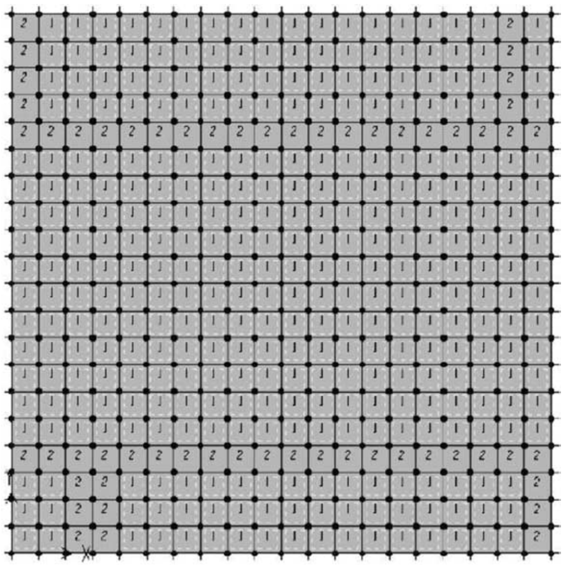

Fig. 4. Finite element model of the test-window 150×150 mm in position D (see Fig. 3): elements type 1 are stone material, type 2 are mortar material.

types, one representing the stone material (phase 1) and the other the mortar material (phase 2). Fig. 4 shows the model of the test-window in position D. It is worth noting that the amount of stone and mortar enclosed by the windows depends on the position of the window, so different values of Reuss's and Voight's limits are obtained for different positions.

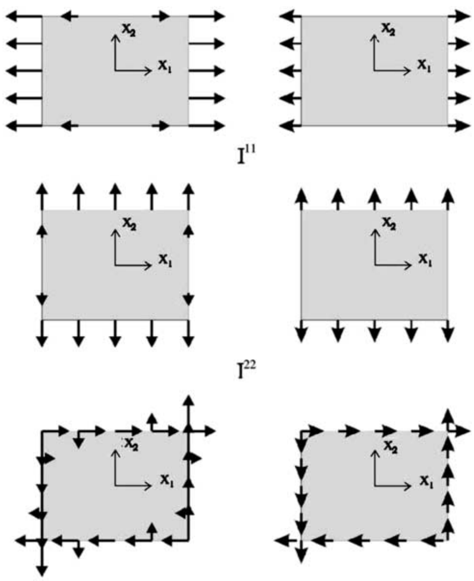

Fig. 5. Boundary conditions of the test-window: (a) *essential* conditions (in terms of nodal displacements); (b) *natural* conditions (in terms of nodal forces).

Table 2
Components of the stiffness tensor (MPa) at the different positions and for increasing size of the test-window (see Fig. 3): (a) test-windows dimensions 150×150 mm; (b) 225×225 mm; (c) 300×300 mm; (d) 375×375 mm

Essential						Natural					Estimates	
ij	Position A	Position B	Position C	Position D	$\langle C^e \rangle$	Position A	Position B	Position C	Position D	$\langle C^n \rangle$	$\widetilde{C}^*$	δ
(a)												
11	6758	9035	10,405	10,390	9147	5693	7985	8960	8570	7802	8475	0.159
12	947	1385	1238	1401	1243	873	1177	1060	1222	1083	1163	0.137
22	5586	7791	5809	7153	6585	5048	6675	4673	6417	5703	6144	0.143
33	4344	6535	5846	6304	5757	3390	4745	3822	4871	4207	4982	0.311
(b)												
11	8377	8980	8545	9506	8852	7463	8237	7867	8500	8017	8434	0.099
12	1250	1166	1124	1456	1249	1118	1082	1046	1311	1139	1194	0.092
22	7223	6267	6189	8214	6973	6511	5719	5781	7652	6416	6695	0.083
33	5702	5297	5139	6484	5656	4594	4367	4203	5465	4657	5156	0.194
(c)												
11	8114	8241	9143	8057	8389	7361	7654	8538	7509	7766	8077	0.077
12	1162	1179	1290	1063	1174	1082	1110	1199	1001	1098	1136	0.066
22	6880	6927	7222	5942	6743	6478	6520	6698	5394	6273	6508	0.072
33	5266	5368	5857	4824	5329	4487	4604	4884	4009	4496	4912	0.170
(d)												
11	7518	7969	8810	8630	8232	6886	7614	8408	8164	7768	8000	0.058
12	1063	1056	1122	1159	1100	1000	1012	1069	1082	1041	1070	0.055
22	6347	6005	6136	6440	6232	5927	5612	5693	6095	5832	6032	0.066
33	4750	4750	5044	5160	4926	4124	4138	4305	4499	4267	4596	0.143

Table 3
Components of the stiffness tensor (MPa) of the entire panel

ij	Essential C^e	Natural C^n	Estimates	
			$\widetilde{C}^*$	δ
11	7992	7889	7940	0.013
12	1023	1016	1020	0.007
22	5691	5596	5644	0.017
33	4149	3912	4030	0.059

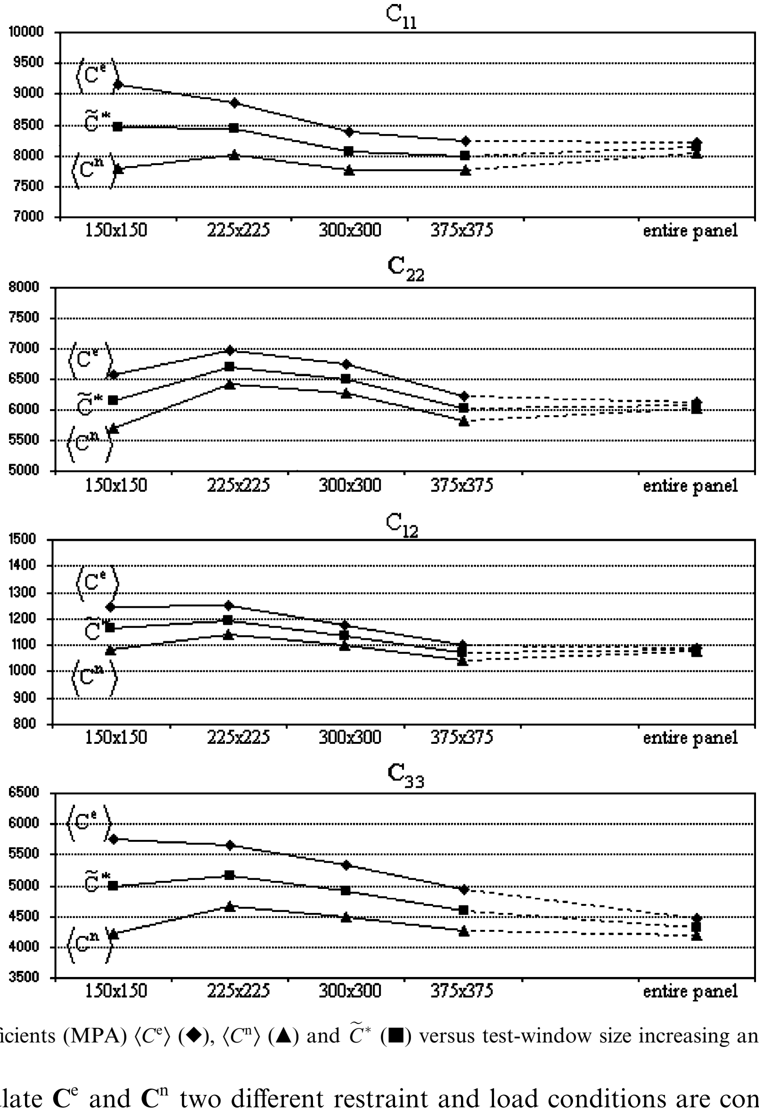

Fig. 6. Stiffness coefficients (MPA) $\langle C^e \rangle$ (◆), $\langle C^n \rangle$ (▲) and $\widetilde{C}^*$ (■) versus test-window size increasing and of the entire panel.

In order to calculate $\mathbf{C}^e$ and $\mathbf{C}^n$ two different restraint and load conditions are considered.

In the first, essential conditions at the boundary in terms of (2) are applied: all the nodes of the boundary are restrained and then assigned displacements. In the second, natural conditions at the boundary in terms of (3) are applied: forces on all the nodes of the boundary are applied.

Furthermore, it is necessary to run the analysis three times for each model, since three different matrices $\mathbf{I}^{mn}$ are required to determine the nine components of $\mathbf{C}^{\mathrm{e}}$ and $\mathbf{C}^{\mathrm{n}}$ (Fig. 5).

Adopting finite elements of the same size (Fig. 4) the mean values of the stresses and the fraction concentration c_2 are immediately calculated. Given the stone and mortar characteristics (Table 1), the $\mathbf{C}^{\mathrm{e}}$ and $\mathbf{C}^{\mathrm{n}}$ stiffness are obtained by means of (15) and (17) respectively.

In this way $\mathbf{C}^{\mathrm{e}}$ and $\mathbf{C}^{\mathrm{n}}$ for the assigned dimensions of test-window and for the different positions of the test-window inside the masonry are available. The averages $\langle\mathbf{C}^{\mathrm{e}}\rangle$ and $\langle\mathbf{C}^{\mathrm{n}}\rangle$ can be calculated, as the estimated stiffness $\widetilde{\mathbf{C}}^{*}$ and the range amplitude δ by means of (18) and (19).

When δ is smaller than a prefixed value δ_{tol}, the value of $\widetilde{\mathbf{C}}^{*}$ can be used as characteristic of the masonry, $\mathbf{C}^{*} = \widetilde{\mathbf{C}}^{*}$. Otherwise, the dimensions of the test-window are increased and new calculations are done.

In the present application, the procedure has been iterated four times, with the dimensions illustrated in Fig. 3 (it should be noted that, although the test-window was increased in size, its position inside the masonry is fixed). The obtained results for essential and natural conditions are reported in Table 2. In order to assess accuracy of the proposed approach, the $\mathbf{C}^{\mathrm{e}}$ and $\mathbf{C}^{\mathrm{n}}$ of the entire masonry panel have been obtained by applying the same boundaries conditions. The results are reported in Table 3.

The estimation of the stiffness components versus the test-window dimensions increasing and of the entire panel (with dimensions: 1657.5×690 mm) are shown in Fig. 6. It should be noted that after four iteration a good convergence is reached.

6. Conclusions

In this paper, a homogenization approach has been proposed to analyze quasi-periodic masonry structures. This approach is based on the concept of RVE which replaces the periodic cell one proposed in literature for periodic pattern.

The RVE is found by using test-window method and by increasing its finite dimensions.

Then the homogenized medium stiffness tensor is obtained by means of the hierarchy of estimates relative to essential and natural boundary conditions with an ensemble average, which is performed on space taking into account different test-window locations on structure. This allows to consider the local heterogeneity of microstructure, increasing the convergence rate and estimates reliability in this way.

A numerical application highlights the effectiveness of the method. In fact, the variation range of the stiffness tensor coefficients was very narrow in limited number of iterations.

The obtained results showed that proposed approach can be used to analyze actual masonry built with blocks having different dimensions, mortar joints having different thickness, which are arranged in a quasi-periodic pattern. Moreover, it can be easily applied to tri-dimensional elastic problem by considering masonry structures with mechanical and geometrical properties varying in thickness.

References

Aboudi, J., 1991. Mechanics of Composite Materials: A Unified Micromechanical Approach. Elsevier, Amsterdam.

Alpa, G., Monetto, I., 1994. Microstructural model for dry block masonry walls with inplane loading. Journal of the Mechanics and Physics of Solids 42 (7), 1159–1175.

Anthoine, A., 1995. Derivation of the in-plane elastic characteristic of masonry through homogenization theory. International Journal of Solids and Structures 32 (2), 137–163.

Bensoussan, A., Lions, J.-L., Papanicolau, G., 1978. Asymptotic Analysis for Periodic Structures. North-Holland, Amsterdam.

Cecchi, A., Di Marco, R., 2000. Homogenization of masonry with a computational oriented procedure. Rigid or elastic block? European Journal of Mechanics, A/Solids 19, 535–546.

Cecchi, A., Sab, K., 2002. A multi-parameter homogenization study for modeling elastic masonry. European Journal of Mechanics, A/Solids 21, 249–268.
Christensen, R.M., 1980. Mechanics of Composite Materials. John Wiley & Sons, New York.
Cuomo, M., Perticone, V., 1996. Determinazione delle proprietà meccaniche della muratura come solido composito: alcuni risultati relativi all'applicazione di metodi di omogeneizzazione. In: Atti del Convegno Nazionale La Meccanica delle Murature tra Teoria e Progetto. Pitagora Editrice, Bologna.
De Buhan, P., De Felice, G., 1997. A homogenization approach to the ultimate strength of brick masonry. Journal of the Mechanics and Physics of Solids 45 (7), 1085–1104.
Eshelby, J.D., 1957. The determination of elastic field of an ellipsoidal inclusion and related problems. Proceedings of Royal Society, A 241, 376–396.
Geymonat, G., Krasuki, F., Marigo, J.-J., 1987. Sur la commutativité des passages à la limite en théorie asymptotique des poutres composites. Comptes Rendus de l'Académie des Sciences Paris, vol. 305, Série II, pp. 225–228.
Giaquinta, M., Giusti, E., 1985. Research on the equilibrium of masonry structures. Archive for Rational Mechanics and Analysis 88, 359–392.
Heymann, J., 1966. The stone skeleton. International Journal of Solids and Structures 3, 249–279.
Hill, R., 1963. Elastic properties of reinforced soils: some technical principles. Journal of the Mechanics and Physics of Solids 11, 357–372.
Huet, C., 1990. Application of variational concepts to size effects in elastic heterogeneous bodies. Journal of the Mechanics and Physics of Solids 38 (6), 813–841.
Luciano, R., Sacco, E., 1997. Homogenization technique and damage model for old masonry material. International Journal of Solids and Structures 34 (24), 3191–3208.
Luciano, R., Sacco, E., 1998a. A damage model for masonry structures. European Journal of Mechanics, A/Solids 17 (2), 285–303.
Luciano, R., Sacco, E., 1998b. Variational methods for the homogenization of periodic heterogeneous media. European Journal of Mechanics, A/Solids 17 (4), 599–617.
Maier, G., Nappi, A., Papa, E., 1991. On damage and failure of brick masonry. In: Donea, J., Jones, P.M. (Eds.), Experimental and Numerical Methods in Earthquake Engineering, The Netherlands, pp. 223–245.
Mori, T., Tanaka, K., 1973. Average stress in the matrix and average elastic energy of materials with misfitting inclusions. Acta Metallurgica 21, 571–574.
Ostoja-Starzewski, M., 1998. Random field models of heterogeneous materials. International Journal of Solids and Structures 35 (19), 2429–2455.
Page, A.W., 1978. Finite element model for masonry. Journal of the Structural Division, Proceedings of the American Society of Civil Engineers 104 (ST8) 1267–1285.
Pande, G.N., Liang, J.X., Middleton, J., 1989. Equivalent elastic moduli for brick masonry. Computers and Geotechnics 8, 243–365.
Pietruszczak, S., Niu, X., 1992. A mathematical description of macroscopic behaviour of brick masonry. International Journal of Solids and Structures 29 (5), 531–546.
Sab, K., 1992. On the homogenization and the simulation of random materials. European Journal of Mechanics, A/Solids 11 (5), 585–607.
Sanchez-Palencia, E., 1980. Non homogeneous media and vibration theory. In: Lecture Notes in Physics, Vol. 127. Springer-Verlag, Berlin.
Sanchez-Palencia, E., Zaoui, A., 1987. Homogenization techniques for composite media. In: Lecture Notes, vol. 272. Springer, Berlin.
Suquet, P.M., 1982. Une méthode duale en homogenénéisation: application aux milieux élastique. Journal de Mécanique Appliquée numéro special, 79–88.